INTEGRALS OF MOTION FOR KICKED QUANTUM SYSTEMS INFN-NA-IV-93/48 DSF-T-93/48

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Abstract:

The generalised quasienergy states are introduced as eigenstates of the new integral of motion for periodically and nonperiodically kicked quantum systems.The photon distribution function of polymode generalised correlated light expressed in terms of multivariable Hermite polynomials is discussed and the relation of its properties to Schrodinger uncertainty relation is given.

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1 Introduction

The nonstationary guantum systems have specific integrals of motion which depend on time in Scrodinger representation (see, for example [1]). The integrals of the motion which are the operators linear in position and momentum and which depend on time in the Schrodinger representation have been constructed for the forced oscillator with time-dependent frequency and for the charged particle moving in time dependent magnetic field in [2] and [3].The linear integrals of the motion for the nonstationary systems with Hamiltonian which is a generic multidimensional quadratic form in positions and momenta operators have been described,for example,in [4]. In all these systems the dependence of the parameters on time was the arbitrary one including both very slow adiabatic case and very fast instant change of parameters.

The important partial case of nonstationary systems is the kicked systems on which the external forces act during very short time in comparison with all the periods of possible vibrations in the system under study.The aim of talk is to discuss properties of the time-dependent invariants for the kicked systems because several interesting phenomena such as extension of Casimir effect to nonstationary conditions or deformations of particle distribution functions may be demonstrated for the kicked systems. The Casimir effect is the phenomenon of attraction of two neutral plates between which is the vacuum of the electromagnetic field due to the dependence of the vacuum energy on the distance between the plates. If one moves the plates the Casimir forces produce the work and due to conservation of the energy the vacuum state becomes the state with photons.So we have nonstationary Casimir effect which is the phenomenon of generation of photons from the vacuum.

The electromagnetic field may be considered as the sum of modes and each mode being the oscillator with its own frequency. When the plates move these oscillators are just the oscillators with time dependent frequencies.So we treat the nonstationary Casimir effect as the effect existing for any system with time dependent parameters.In this case the frequences are these parameters (like distance between the plates).Thus the linear invariants found in [2] and [4] for one dimensional and multidimensional oscillators are in the case of nonstationary Casimir effect the integrals of motion for electromagnetic field with time varying boundaries.

Quantum systems with energy spectra are the systems with stationary Hamiltonians.The dynamics of these systems is described by the transitions between the energy levels.The nonstationary quantum systems have no energy levels.But for periodical quantum systems the notion of quasienergy levels has been introduced in [5] and [6].The main point of the quasienergy concept is to relate the quasienrgies to the eigenvalues of the Floquet operator which is equal to the evolution operator of a quantum system taken at a given time moment.

In [7] the connection of Floquet operator and quasienergies with time-dependent integrals of motion has been found. Following this article we will discuss the relation of the Floquet operator to integrals of motion and introduce a new operator which is the integral of motion and has the same quasienergy spectrum that the Floquet operator has.Implicitly this result was contained in [4].

Thus,the quasienergy of the periodically kicked quantum systems may be shown to be the time-dependent integral of motion for such systems.If the system is kicked nonperiodically it is possible nevertherless to find out the time-dependent integrals of motion and to introduce the generalized quasienergy states [7]. It gives the possibility to extend the analisys of quantum chaos phenomenon which is usually based on the studying the quasienergy spectra of the kicked systems to the case of nonperiodically kicked systems.

Another physical phenomenon related to the kicking is the creating the photons in squeezed states from vacuum due to the nonstationary Casimir effect mentioned above. We will discuss how the electromagnetic field (or pion or boson field of any kind) changes their statistical properties due to influence of the kicking.As an example the state with the Wigner function of generic Gaussian form will be considered as the final result of kicking.

2 Generalized quasienergies

Following [7] we will discuss the properties of quasienergies and generalized quasienergies for periodically and nonperiodically kicked quantum systems.

If one has the system with hermitian Hamiltonian $H(t)$ such that $H(t+T)=H(t)$ the unitary evolution operator $U(t)$ is defined as follows

$$|\psi,t>=U(t)|\psi,0> \tag{1}$$

where $|\psi,0>$ is a state vector of the system taken at the initial time moment.Then by definition the operator $U(T)$ is called the Floquet operator and its eigenvalues have the form

$$f=exp(-iET) \tag{2}$$

where E is called the quasienergy and the corresponding eigenvector is called the quasienergy state vector.The spectra of quasienergy may be either discrete or continuous ones (or mixed) for different quantum systems. For multidimensional systems with quadratic Hamiltonians the quasienergy spectra have been related to real symplectic group $ISp(2N,R)$ and found explicitly in [4].

We want to answer the following questions. Is the Floquet operator $U(T)$ the integral of motion?The operator $U(T)$ does not satisfy the relation

$$dI(t)/dt+i[H(t),I(t)]=0,(\hbar=1) \tag{3}$$

which defines the integral of motion $I(t)$.So,the Floquet operator $U(T)$ is not the integral of motion for the periodical nonstationary quantum systems.But as it was pointed out in [4] any operator of the form

$$I(t)=U(t)I(0)U^{-1}(t) \tag{4}$$

satisfies the equation (3) and this operator is the integral of motion for the quantum system under study.Let us apply this ansatz to the case of periodical quantum systems.We introduce the unitary operator $M(t)$ which has the form

$$M(t)=U(t)U(T)U^{-1}(t). \tag{5}$$

This operator is the integral of motion due to the construction given by the formula (4) for any integral of motion.The spectrum of the new invariant operator $M(t)$ coincides with the spectrum of the Floquet operator $U(T)$.We have proved that since quasienergies are defined as eigenvalues of the integral of motion $M(t)$ they are conserved quantities.

The given construction permits us to introduce new invariant labels for nonperiodical systems,for example, with the time-dependence of the Hamiltonian corresponding to quasicrystal structure in time described by two (or more) characteristic times.For such systems the analog of the invariant Floquet operator (5) will be given by the relation

$$M_1(t) = U(t)U(t_1)U(t_2)U^{-1}(t). \tag{6}$$

The eigenvalues of the operator $M_1(t)$ are the consrved quantities and they characteryse the nonperiodical quantum systems in the same manner as quasienergies describe the states of periodical quantum systems. This operator determines the generalized quasienergies and its obvious analog describes the geometrical phase which is the characteristic of special nonperiodical system for which the parameters of the Hamiltonian take their initial values after some time T [8].

The quasienergy spectrum of periodically kicked quantum systems may be connected with quantum chaos phenomenon (see [9], [10]).In [11] the integral of motion for delta-kicked nonlinear oscillator has been found to exist even in the case of chaotic behaviour.In [12] the symmetry group criterium for the periodically delta-kicked systems has been found to obtain either regular or chaotic behaviour of these systems.The criterium relates the Floquet operator spectrum to the conjugacy classes of the system symmetry group.The results of [12] may be applied to the nonperiodical systems too. The generalized quasienergy spectrum is determined by the conjugacy class of the same group to which belongs the integral of motion (6).For quadratic systems it is the same real symplectic group $Sp(2N, R)$.In the case of two characteristic times the classification of the possible either chaotic or regular regimes of the system under study coincides formally with the classification for the delta-kicked periodical quantum systems given in [12].

The geometrical phase is defined as the phase of eigenvalue of the evolution operator $U(T)$ where T is the time moment at which the parameters of Hamiltonian take their initial values.Using similar arguments we can answer the same question as for the quasienergies.Is the geometrical phase the integral of motion of the nonstationary and nonperiodical quantum system?The answer is "yes" because such nonperiodical system has characteristic time T. Due to this the operator $U(t)U(T)U^{-1}(t)$ is the integral of motion of the system.

From the point of view of given analisys the nonperiodical systems with several characteristic times have to demonstrate similar physical properties that are usually considered as properties of purely periodical quantum systems.Thus,the types of chaotic and regular regimes of the kicked quantum systems have to be the same for both periodical and nonperiodiacal kicks and group classification of regular and irregular behaviour of kicked systems given in [12] may be easily extended to the case of nonperiodically kicked systems.

3 Photon distributions

Let us consider as nonstationary quantum system the photons in a resonator.We want to discuss the photon distribution function which is the distribution for the generalized correlated state found in [13].Below we follow this work. Due to the varying boundaries and time dependence of the refraction index of the media in a resonator the mixed squeezed state of the N-mode light with density operator $\hat{\rho}$ may emerge.It is described by Wigner function $W(\mathbf{p},\mathbf{q})$ of the generic Gaussian form which contains $2N^2+3N$ real parameters.$2N$ parameters are mean values of light quadratures $<\mathbf{p}>$ and quadratures $<\mathbf{q}>$ and other $2N^2+N$ parameters are matrix elements of the real symmetric dispersion matrix m with four N-dimensional block matrices three of which are

$$m_{11}=\sigma_{\mathbf{p}}, \tag{7}$$
$$m_{12}=\sigma_{\mathbf{pq}}, \tag{8}$$
$$m_{22}=\sigma_{\mathbf{q}}. \tag{9}$$

Below we will use the invariant parameters

$$T=Tr\ m \tag{10}$$

and

$$d=\det m. \tag{11}$$

Also we will use the other invariant coefficients of the polynomial

$$P(x)=\det(m-x\mathbf{1}) \tag{12}$$

where $\mathbf{1}$ is $2N$-dimensional identity matrix.

We will introduce the notations

$$\mathbf{Q}=(\mathbf{p},\mathbf{q}) \tag{13}$$

where $2N$-dimensional vector $\mathbf{Q}$ consists of N components of light quadratures $p_1,...,p_N$ and N components of quadratures $q_1,...,q_N$. The generic gaussian Wigner function has the form (see,for example,[1])

$$W(\mathbf{p},\mathbf{q})=d^{-\frac{1}{2}}\exp[-(2)^{-1}[(\mathbf{Q}-<\mathbf{Q}>)m^{-1}(\mathbf{Q}-<\mathbf{Q}>)]. \tag{14}$$

The parameters $<\mathbf{p}>$ and $<\mathbf{q}>$ are given by the formulae

$$<\mathbf{p}>=Tr\ \hat{\rho}\hat{\mathbf{p}}, \tag{15}$$
$$<\mathbf{q}>=Tr\ \hat{\rho}\hat{\mathbf{q}}, \tag{16}$$

where the operators $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ are the quadrature components of photon creation $\mathbf{a}\dagger$ and the annihilation $\mathbf{a}$ operators

$$\hat{\mathbf{p}}=\frac{\mathbf{a}-\mathbf{a}\dagger}{i\sqrt{2}}, \tag{17}$$
$$\hat{\mathbf{q}}=\frac{\mathbf{a}+\mathbf{a}\dagger}{\sqrt{2}}. \tag{18}$$

Due to the physical meaning of the dispersions the diagonnal elements of the matrix m must be nonnegative numbers,so the invariant parameter T Eq.(10) is a positive number.Also the determinant d Eq.(11) of the dispersion matrix must be positive.In fact the matrix m must be positive-definite.

To obtain the photon distribution function we have to calculate the probability $P_{\mathbf{n}}$ to have $\mathbf{n}$ photons in the state with the density operator $\hat{\rho}$. Here the vector $\mathbf{n}$ has N components n_i which are nonnegative integers. This probability is given by the formula

$$P_{\mathbf{n}} = Tr\ \hat{\rho}|\mathbf{n}><\mathbf{n}|, n_i = 0, 1, 2, ... \quad i = 1, 2, ..., N \tag{19}$$

where the number states $|\mathbf{n}>$ are the eigenstates of the number operator $\mathbf{a}\dagger\mathbf{a}$

$$\mathbf{a}\dagger\mathbf{a}|\mathbf{n}>=\mathbf{n}|\mathbf{n}>. \tag{20}$$

The function $P_{\mathbf{n}}$ may be obtained if one calculates the generating function for the matrix elements $\rho_{\mathbf{mn}}$ of the density operator $\hat{\rho}$ in the Fock basis. This generating function is the matrix element of the density operator in the coherent state basis

$$<\beta|\hat{\rho}|\alpha>= \exp(\frac{-|\alpha|^2}{2} - \frac{|\beta|^2}{2}) \sum_{\mathbf{m},\mathbf{n}=\mathbf{0}}^{\infty} \frac{(\beta^*)^{\mathbf{m}}\alpha^{\mathbf{n}}}{(\mathbf{m}!\mathbf{n}!)^{\frac{1}{2}}}\rho_{\mathbf{mn}} \tag{21}$$

Here and below we use notations:$\alpha$and β are N-dimensional vectors with complex components and

$$\mathbf{n}! = n_1!n_2!...n_N!, \tag{22}$$
$$\alpha^{\mathbf{n}} = \alpha_1^{n_1}\alpha_2^{n_2}...\alpha_N^{n_N} \tag{23}$$

and

$$\sum_{\mathbf{m},\mathbf{n}=\mathbf{0}}^{\infty} = \sum_{m_1=0}^{\infty} ... \sum_{m_N=0}^{\infty}\sum_{n_1=0}^{\infty} ... \sum_{n_N=0}^{\infty}. \tag{24}$$

We have

$$P_{\mathbf{n}} = \rho_{\mathbf{nn}}. \tag{25}$$

The N-mode coherent state $|\alpha>$ is the normalized eigenstate of the annihilation operator

$$\mathbf{a}|\alpha>=\alpha|\alpha> . \tag{26}$$

In terms of Wigner function the density operator in the coherent state representation has the form of $2N$-dimensional overlap integral [1]

$$<\beta|\hat{\rho}|\alpha>= \frac{1}{(2\pi)^N}\int W(\mathbf{p},\mathbf{q})W_{\alpha\beta}(\mathbf{p},\mathbf{q})d\mathbf{p}d\mathbf{q}, \tag{27}$$

where the function $W_{\alpha\beta}(\mathbf{p},\mathbf{q})$ is the Wigner function of the operator $|\alpha><\beta|$.It has the form [1]

$$W_{\alpha\beta}(\mathbf{p},\mathbf{q}) = 2^N \exp[-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} - \alpha\beta^* - \mathbf{p}^2 - \mathbf{q}^2 + \sqrt{2}\alpha(\mathbf{q} - i\mathbf{p}) + \sqrt{2}\beta^*(\mathbf{q} + i\mathbf{p})]. \tag{28}$$

Let us introduce the $2N$-dimensional vector

$$\gamma = (\beta^*, \alpha) \tag{29}$$

which is composed from two N-dimensional vectors.Let us introduce also $2N$- dimensional unitary matrix u which has four N-dimensional block matrices

$$u_{11} = -u_{12} = -iu_{21} = -iu_{22} = -\frac{i}{\sqrt{2}}\mathbf{1} \tag{30}$$

and $\mathbf{1}$ is N-dimensional identity matrix.

Since the integral given by Eq.(27) is the Gaussian one it may be easily calculated. So,we have

$$< \beta|\hat{\rho}|\alpha >= P_0 \exp(-\frac{|\gamma|^2}{2}) \exp[-\frac{1}{2}\gamma R\gamma + \gamma R\mathbf{y}] \tag{31}$$

where the symmetric $2N$-dimensinal matrix R has the matrix elements expressed in terms of the dispersion matrix m as follows

$$R = -2u^{-1}(\frac{m^{-1}}{2} + 1)^{-1}u^* + \Sigma_x \tag{32}$$

The matrix Σ_x is $2N$-dimensional analog of Pauli matrix. The $2N$-dimensional vector $\mathbf{y}$ is given by the relation

$$\mathbf{y} = 2u^t(1 - 2m)^{-1} < \mathbf{Q} > \tag{33}$$

where the matrix u^t is transposed one and the factor P_0 has the form

$$P_0 = [det(m + \frac{1}{2})]^{-\frac{1}{2}} \exp[- < \mathbf{Q} > (2m + 1)^{-1} < \mathbf{Q} >] \tag{34}$$

We obtain for the photon distribution function P_n the expression

$$P_{\mathbf{n}} = P_0 \frac{H^{\{R\}}_{\mathbf{nn}}(\mathbf{y})}{\mathbf{n}!}. \tag{35}$$

Here the matrix R determining the Hermite polynomial is given by the formulae (32) and argument of Hermite polynomial is given by the expression (33).The expression (35) is the partial case of the matrix elements of the density operator in Fock states basis obtained in [14] by canonical transform method.

The behaviour of the distributions may be wavy function of photon numbers as it is in one-mode case [15],[16].

Mean values of photons in each mode have the form [13]

$$< n_j >= \frac{1}{2}(\sigma_{p_jp_j} + \sigma_{q_jq_j} - 1) + \frac{1}{2}(< p_j >^2 + < q_j^2). \tag{36}$$

Photon number variances are given by expression

$$\sigma_{n_j} = \frac{1}{2}(T_j^2 - 2d_j - \frac{1}{2}) + < Q_j > m_j < Q_j > \tag{37}$$

where T_j and d_j are the trace and the determinant of the photon quadrature 2x2-dispersion matrix m_j of the j-th mode only and the 2-vector Q_j has components p_j, q_j. The correlations of photon numbers in different modes may be expressed analogously.

4 Uncertainty relation and distributions

Following [17] for one-mode case we consider the photon distribution function (35) expressed in terms of the series of the products of two Hermite polynomials.

There exists the expression of the Hermite polynomial of two variables with equal indicies in terms of the products of two Hermite polynomials of one variable[14]. Using this formula one obtains from the general expression (35) the following photon distribution function

$$P_n = P_0 n! \sum_{k=0}^{n} (\frac{R_{11}R_{22}}{4})^{n/2} (-\frac{2R_{12}}{\sqrt{R_{11}R_{22}}})^k (n-k)!^{-2} k!^{-1} |H_{n-k}(\frac{R_{11}y_1 + R_{12}y_2}{\sqrt{2R_{11}}})|^2. \qquad (38)$$

Here the factor P_0 is given by the formula (34),the matrix elements of the matrix R are given by the formulae (32) and the numbers y are given by the formula (33).If the state is the pure squeezed and correlated state the determinant of the quadrature dispersion matrix is equal to $\frac{1}{4}$ and the matrix element $R_{12} = 0$ due to the formula (32).In this case only the first term in the series is not equal to zero and this term gives the known expression for the photon distribution function of the pure and correlated state obtained by another method in [16]. The above expression is convinient to discuss the connection of the uncertainty relations with the quantum distribution functions.In fact the formula for the Gaussian Wigner function (14) formally coincides with the classical Gaussian density of probability in the particle phase space.Then we have to answer the question in what aspects these formulae must be considered as essentially different ones.By intuition it is clear that the uncertainty relation must distinguish the classical and quantum Gaussian densities of probabilities having absolutely the same form (14). And in fact in addition to the usual restriction for the dispersion matrix for the Gaussian classical distribution function in the phase space which is the condition of nonnegativity of this dispersion matrix the quantum mechanics demands the inequality for the determinant of this matrix $d > 1/4$.This inequality is the Schrodinger uncertainty relation.The formula (38) permits to relate this inequality with physically obvious condition that probability to find n photons must be nonnegative because the determinant of the dispersion matrix is the parameter on which depends the photon distribution function.We see that all the terms in the expression (38) are obviously positive ones except the term containing the number $-2R_{12}$ which may change the sign for odd powers k if it is not positive itself.It means that for natural condition of positiveness of photon distribution function it is nesessary to have inequality

$$R_{12} < 0 \qquad (39)$$

But this inequlity is equivalent to the mentioned above inequality for the dispersion matrix determinant d which implies the uncertainty relation.Thus existence of the connection of the Wigner function (14) with the photon distribution funtion (38) is consistent only if the uncertainty relation holds.So we clarify the mechanism how uncertainty relation in phase space of electromagnetic field oscillator influences the form of photon distribution function.

As we discussed the nonstationary Casimir effect produces the changes in statistical properties of the photons.For example,it changes Poisson distribution of photons to become the discussed distribution expressed in terms of multivariable Hermite polynomials. Also it deforms Planck distribution to become [14]

$$\bar{n} = \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1} + |v|^2 \coth\frac{\hbar\omega}{2kT} + |\delta|^2, \qquad (40)$$

where the correction to usual Planck distribution term contains parameters $|v|^2$ and $|\delta|^2$.These parameters depend on the characteristics of kicking [1],[14].So,a generic kicking produces from ground state the squeezed mixed state with deformed Planck distribution and the number state distribution function which is expressed explicitly in terms of Hermite polynomials of several variables.

It is interesting to note that if to take into account a possible nonlinearity of classical electrodynamics we can have another reason for deforming photon distribution formulae. In [18] it was suggested that the possible nonlinearity of the electromagnetic field vibrations which must exist for very large amplitudes corresponding to high densities of the field energy may be considered as the nonlinearity described by the q-oscillator. It was seen that it is subject to nonlinear vibrations with a special kind of dependence of the frequency on the amplitude and the influence of such nonlinearity on Bose distribution function was evaluated. So,for the thermal state we have the deformed Planck distribution formula [18] of the form

$$(\bar{n})_q \simeq \frac{1}{e^{\frac{\hbar\omega}{kT}}-1} - \kappa^2 \frac{\hbar\omega}{kT} \frac{e^{\frac{3\hbar\omega}{kT}} + 4e^{\frac{2\hbar\omega}{kT}} + e^{\frac{\hbar\omega}{kT}}}{(e^{\frac{\hbar\omega}{kT}}-1)^4}. \tag{41}$$

Here the first term is the usual Planck distribution formula and its correction is proportional to the square of the nonlinearity parameter κ.
It means that black body radiation formula changes due to the nonlinearity of the electromagnetic field vibrations.So nonstationary Casimir effect and q-nonlinearity of electromagnetic field vibrations produce deformations of Planck distribution formula.But the temperature dependence of corrections to Planck formula is different and it gives a possibility to distinguish the influence of these effects.

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